FREE TRIAL

SEQUOIA
— EDUCATION —

Specialist Mathematics Examination 1

Solutions Book

VCE Trial Examination – Free

Contents pages

© Sequoia Education 2025

Question 1 (3 marks)

MARK 1. Differentiates expression implicitly
MARK 2. Rearranges for $\frac{dy}{dx}$, or equivalent merit
MARK 3. Provides correct answer

Using the product rule and chain rule,

$$\begin{aligned}\left(y+x\frac{dy}{dx}\right)e^{xy}=2y\frac{dy}{dx} &\implies (2y-xe^{xy})\frac{dy}{dx}=ye^{xy}\\ &\implies \frac{dy}{dx}=\frac{ye^{xy}}{2y-xe^{xy}}\\ &\implies \left.\frac{dy}{dx}\right|_{(0,2)}=\frac{2}{4-0}=\frac{1}{2}.\end{aligned}$$

Question 2a (1 mark)

MARK 1. Provides correct answer

Using a Pythagorean identity, we have

$$2x=\cot(t),\ \ y+2=\csc(t) \implies (y+2)^2-4x^2=1.$$

Question 2b (3 marks)

MARK 1. Labels axial intercepts
MARK 2. Labels asymptotes
MARK 3. Shows correct graph shape

When $y=0$, we get

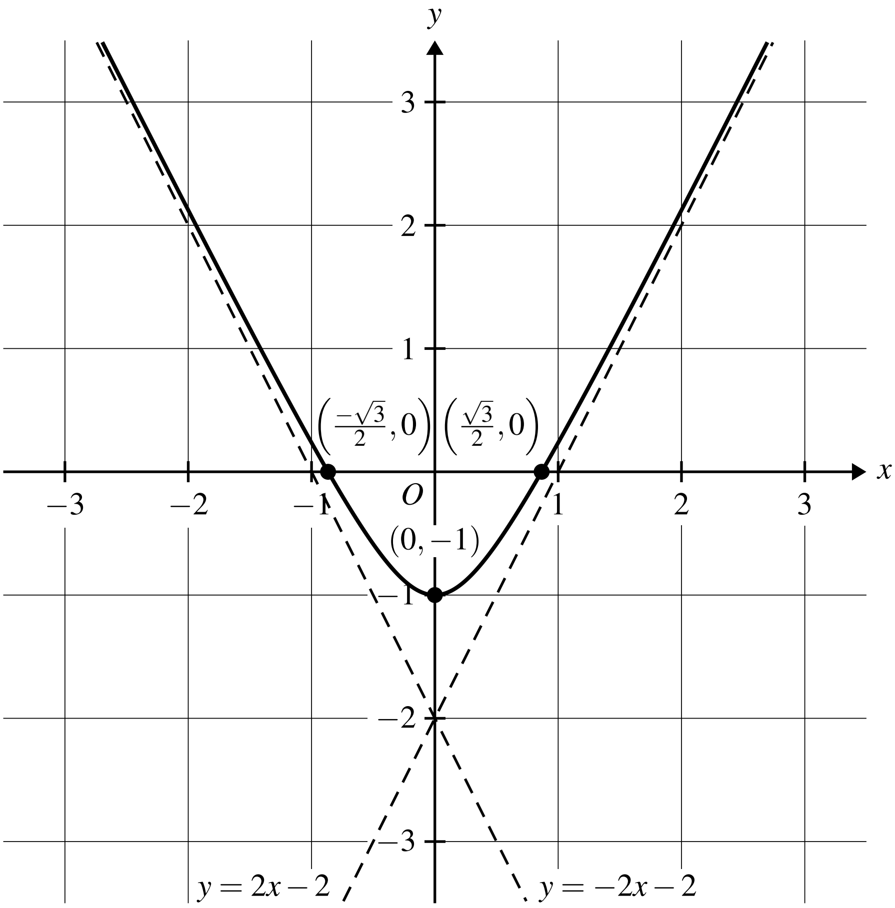
$$4-4x^2=1 \implies x=\pm\frac{\sqrt{3}}{2}.$$

In standard form, the hyperbola is given by

$$\frac{(y+2)^2}{1^2}-\frac{x^2}{\left(\frac{1}{2}\right)^2}=1,$$

and so the asymptotes are given by $y=2x-2$ and $y=-2x-2$. Also note the restriction $0<t<\pi$.

(Graph is shown on the following page.)



Question 3a (1 mark)

MARK 1. Shows sufficient and correct work to arrive at conclusion

Working from the right-hand side, we get

$$2\operatorname{cis}\left(-\frac{\pi}{6}\right) = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i.$$

Question 3b (2 marks)

MARK 1. Applies de Moivre's theorem, or equivalent merit
MARK 2. Provides correct answer

By de Moivre's theorem, we have

$$\left(\sqrt{3} - i\right)^n = 2^n \operatorname{cis}\left(-\frac{n\pi}{6}\right).$$

So, $\left(\sqrt{3} - i\right)^n \in \mathbb{R}$ if and only if

$$\operatorname{Im}\left(\left(\sqrt{3} - i\right)^n\right) = 2^n \sin\left(-\frac{n\pi}{6}\right) = 0 \iff n = 6k, \quad k \in \mathbb{Z}.$$

Question 4a (2 marks)

MARK 1. Constructs confidence interval
MARK 2. Provides correct answer

A 95% confidence interval is given by

$$\text{CI} = \left(350 - 2 \cdot \frac{72}{\sqrt{256}},\ 350 + 2 \cdot \frac{72}{\sqrt{256}}\right) = (350 - 9,\ 350 + 9) = (341, 359).$$

Question 4b (2 marks)

MARK 1. Establishes inequation involving margin of error, or equivalent merit
MARK 2. Provides correct answer

We have that

$$2 \cdot \frac{72}{\sqrt{n}} \le 4 \implies \frac{1}{\sqrt{n}} \le \frac{1}{36} \implies n \ge 1296.$$

That is, the smallest such sample size is 1296.

Question 5 (4 marks)

MARK 1. Finds $\overrightarrow{BA}$ and $\overrightarrow{BC}$
MARK 2. Applies geometric interpretation of dot product, or equivalent merit
MARK 3. Establishes quadratic equation for λ, or equivalent merit
MARK 4. Provides correct answer

First, we have that $\overrightarrow{BA} = (\lambda - 1)\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{BC} = -\underset{\sim}{i} + \underset{\sim}{k}$. Therefore,

$$\begin{aligned}
\overrightarrow{BA} \cdot \overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right| \cos\left(\frac{2\pi}{3}\right) &\implies 2 - \lambda = \sqrt{2} \cdot \sqrt{(\lambda - 1)^2 + 2} \cdot \frac{-1}{2} \\
&\implies 2\lambda - 4 = \sqrt{2(\lambda - 1)^2 + 4} \qquad (\dagger) \\
&\implies 4\lambda^2 - 16\lambda + 16 = 2\lambda^2 - 4\lambda + 6 \\
&\implies (\lambda - 5)(\lambda - 1) = 0
\end{aligned}$$

Observe that in equation (†), we require that $2\lambda - 4 \ge 0$, so $\lambda = 5$.

Question 6a (1 mark)

MARK 1. Applies appropriate constant acceleration formula to show conclusion

Using the constant acceleration formula $v = u + at$ with $u = 0$, $a = g$ and $t = 1$ gives

$$v = 0 + g \cdot 1 = g \text{ m s}^{-1}.$$

Question 6b (4 marks)

MARK 1. Applies separation of variables, or equivalent merit
MARK 2. Establishes general solution for speed by antidifferentiation, or equivalent merit
MARK 3. Finds constant of integration, or equivalent merit
MARK 4. Provides correct answer

Let V be the speed of the skydiver after six seconds. Since $v(1) = g$, by separation of variables, we have

$$\int_g^V \frac{5}{5g - v}\, dv = \int_1^6 dt \implies \left[-5\log_e(|5g - v|)\right]_g^V = 5.$$

Because of the initial condition, the absolute value brackets can be dropped, and so

$$5 = -5\log_e(5g - V) + 5\log_e(4g) = 5\log_e\left(\frac{4g}{5g - V}\right).$$

Rearranging for V, we get

$$\frac{4g}{5g - V} = e \implies V = \left(5 - \frac{4}{e}\right) g \text{ m s}^{-1}.$$

Question 6c (1 mark)

MARK 1. Provides correct answer

As $a \to 0^+$, we have $v \to 5g = 49$ m s^{-1}, so the terminal speed of the skydiver is 49 m s^{-1}.

Question 7 (4 marks)

MARK 1. Verifies base case
MARK 2. States induction hypothesis, or equivalent merit
MARK 3. Differentiates induction hypothesis for induction step
MARK 4. Shows sufficient and correct algebraic work throughout to arrive at conclusion

For the base case $n = 1$, observe that

$$\frac{d^1}{dx^1}\left(\frac{1}{x}\right) = -\frac{1}{x^2} = (-1)^1 \frac{1!}{x^{1+1}}.$$

Now, let $k \in \mathbb{Z}^+$ be arbitrary and suppose that the statement holds for $n = k$. Then,

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(\frac{d^k}{dx^k}\left(\frac{1}{x}\right)\right) = \frac{d}{dx}\left((-1)^k \frac{k!}{x^{k+1}}\right) = -(-1)^k \frac{k!(k+1)}{x^{k+2}} = (-1)^{k+1} \frac{(k+1)!}{x^{(k+1)+1}}.$$

Thus, by the principle of mathematical induction, the statement is true.

Question 8 (5 marks)

MARK 1. Differentiates y with respect to x
MARK 2. Expresses arc length as integral
MARK 3. Simplifies integrand to remove square root
MARK 4. Antidifferentiates resulting integrand
MARK 5. Provides correct answer

Differentiating y with respect to x gives

$$\frac{dy}{dx} = \frac{-2x}{1-x^2},$$

and so the length of the arc is given by

$$L = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2}\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{x^4+2x^2+1}{(1-x^2)^2}}\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2+1}{1-x^2}\,dx,$$

noting that $\frac{x^2+1}{1-x^2} > 0$ for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Using the given fact, we therefore get

$$L = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{1+x} + \frac{1}{1-x} + 1\right) dx = \Big[\log_e(|1+x|) - \log_e(|1-x|) + x\Big]_{-\frac{1}{2}}^{\frac{1}{2}}.$$

Evaluating this gives

$$L = \log_e\left(\frac{3}{2}\right) - \log_e\left(\frac{1}{2}\right) + \frac{1}{2} - \left(\log_e\left(\frac{1}{2}\right) - \log_e\left(\frac{3}{2}\right) - \frac{1}{2}\right) = \log_e(9) + 1.$$

Question 9a (2 marks)

MARK 1. Applies double angle formula, or equivalent merit
MARK 2. Shows sufficient and correct algebraic work to arrive at conclusion

Using the double angle formula for cosine, we get

$$\frac{1-t^2}{1+t^2} = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)} = \frac{\cos(x)}{1} = \cos(x).$$

Question 9b (3 marks)

MARK 1. Differentiates t with respect to x
MARK 2. Substitutes terminals of integration
MARK 3. Simplifies integrand to arrive at conclusion

Differentiating t with respect to x gives

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(\tan^2\left(\frac{x}{2}\right)+1\right) = \frac{1}{2}\left(t^2+1\right) \implies \frac{dx}{dt} = \frac{2}{t^2+1}.$$

Moreover, the substitution gives $t(0) = 0$ and $t\left(\frac{\pi}{2}\right) = 1$. Hence,

$$\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos(x)}\,dx = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int_0^1 \frac{2}{2\left(t^2+1\right)+1-t^2}\,dt = \int_0^1 \frac{2}{t^2+3}\,dt.$$

Question 9c (2 marks)

MARK 1. Antidifferentiates integrand
MARK 2. Provides correct answer

By **part b.** and the formula sheet, we have

$$\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos(x)}\,dx = \int_0^1 \frac{2}{t^2+3}\,dt = \left[\frac{2}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)\right]_0^1.$$

Evaluating this gives

$$\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos(x)}\,dx = \frac{2}{\sqrt{3}}\left(\frac{\pi}{6}-0\right) = \frac{\pi\sqrt{3}}{9}.$$

End of Solutions Book

© Sequoia Education 2025